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A Note on $f''' + ff'' + \lambda(1 - f'^2) = 0$ with $\lambda \in \left(-\frac{1}{2}, 0\right)$ Arising in Boundary Layer Theory

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Abstract—For any $\lambda \in (-1/2, 0)$, there exists $f(\eta) \in C^1[0, +\infty)$ such that

$$f''' + ff'' + \lambda(1 - f'^2) = 0, \qquad \text{a.e. in } (0, +\infty), \quad f(0) = 0, \quad f'(0) = 0, \quad f'(+\infty) = 1,$$

which arises in boundary layer theory in fluid mechanics.

Keywords—Boundary value problems, Fixed points, Existence results.

1. INTRODUCTION AND PRELIMINARIES

Falkner and Skan [1] first deduced the following third-order nonlinear differential equation for $f(\eta)$:

$$f''' + ff'' + \lambda\left(1 - f'^2\right) = 0, \qquad 0 < \eta < +\infty, \tag{1.1}$$

with boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(+\infty) = 1, \tag{1.2}$$

which arises in boundary layer theory in fluid mechanics, and is of great importance in the boundary layer theory in fluid mechanics [2].

About problem (1.1),(1.2), Wang, Gao and Zhang [3] studied the case of $\lambda \leq -1/2$ and $\lambda \geq 0$ by the following nonlinear second-order singular boundary value problem (1.3),(1.4):

$$w''(t) = -\lambda\left(\frac{1 - t^2}{w(t)}\right)' - \frac{t}{w(t)}, \qquad 0 \leq t < 1, \tag{1.3}$$

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with boundary conditions

$$w'(0)w(0) = -\lambda, \qquad w(1) = 0. \tag{1.4}$$

Recently, Yang [4] proved by the study of (1.3),(1.4) that there exists $\lambda_0 \in (-1/2, 0)$ such that problem (1.1),(1.2) has a solution $f(\eta) \in C^2[0, +\infty)$ at least for any fixed $\lambda \in (\lambda_0, 0)$.

For the case of $\lambda \in (-1/2, \lambda_0]$, as far as we know, very few results are available for (1.1),(1.2) and it is significant to determine the existence of $C^1[0, +\infty)$ solutions to (1.1),(1.2) [2, p. 292].

By the study of the related integral equations, we provide in this paper some existence information of $C^1[0, +\infty)$ solutions to problem (1.1),(1.2), i.e., for any fixed $\lambda \in (-1/2, \lambda_0]$, there exists $f(\eta) \in C^1[0, +\infty)$ to satisfy

$$f''' + ff'' + \lambda\left(1 - f'^2\right) = 0, \qquad \text{a.e. in } (0, +\infty), \quad f(0) = 0, \quad f'(0) = 0, \quad f'(+\infty) = 1. \tag{$*$}$$

The results of this paper will fill the gap in the literature.

Throughout this paper, suppose that $\lambda \in (-1/2, 0)$, then $s_0 = -\lambda/(1 + \lambda) \in (0, 1)$, $-\lambda(1 - s^2) - (1 - s)s = (1 - s)(-\lambda(1 + s) - s) = (\lambda + 1)(1 - s)(s_0 - s) \geq 0$, $0 \leq s \leq s_0$.

LEMMA 1.1. *There exists $\tilde{w}_0(t) \in C[0, s_0]$ such that*

$$\tilde{w}_0(t) = \int_0^t \frac{-\lambda(1 - s^2) - s(t - s)}{\tilde{w}_0(s)}\, ds + 1, \qquad 0 \leq t \leq s_0. \tag{1.5}$$

PROOF. Put $C = \{w(t) : w(t) \in C[0, s_0],\ 1 \leq w(t) \leq 2\}$. For $w(t) \in C$, define an operator as follows:

$$Tw(t) = \int_0^t \frac{-\lambda(1 - s^2) - s(t - s)}{w(s)}\, ds + 1.$$

Notice $1 \geq -\lambda(1 - s^2) - (t - s)s \geq -\lambda(1 - s^2) - (1 - s)s \geq 0$, $0 \leq s \leq t \leq s_0$ and $Tw(t) \leq \int_0^t ds + 1 \leq 2$, we know that T is an operator from C into C.

Let $w(t), w_1(t) \in C$,

$$\begin{aligned} |Tw(t) - Tw_1(t)| &= \left| \int_0^t \left[\frac{-\lambda(1 - s^2) - s(t - s)}{w(s)} - \frac{-\lambda(1 - s^2) - s(t - s)}{w_1(s)} \right] ds \right| \\ &\leq \int_0^{s_0} \left| \frac{(-\lambda(1 - s^2) - s(t - s))(w_1(s) - w(s))}{w(s)w_1(s)} \right| ds \leq \|w - w_1\|, \end{aligned}$$

i.e., T is continuous.

For $w(t) \in C$, $t', t'' \in [0, s_0]$, $t' < t''$,

$$\begin{aligned} |Tw(t'') - Tw(t')| &= \left| \int_0^{t''} \frac{-\lambda(1 - s^2) - s(t'' - s)}{w(s)}\, ds - \int_0^{t'} \frac{-\lambda(1 - s^2) - s(t' - s)}{w(s)}\, ds \right| \\ &= \left| \int_{t'}^{t''} \frac{-\lambda(1 - s^2) - s(t'' - s)}{w(s)}\, ds + \int_0^{t'} \frac{(t' - t'')s}{w(s)}\, ds \right| \leq 2\left|t'' - t'\right|, \end{aligned}$$

which implies that T is compact. The Schauder fixed-point theorem tells us that Lemma 1.1 holds. This completes the proof.

LEMMA 1.2. *Let $\tilde{w}_0(t)$ be as in Lemma 1.1, then there exists $\tilde{w}_1(t) \in C[s_0, 1]$ such that*

$$\tilde{w}_1(t) = \int_t^1 \frac{(1 - s)(\lambda + \lambda s + s)\, ds}{\tilde{w}_1(s)} + (1 - t)\left[\int_{s_0}^t \frac{s\, ds}{\tilde{w}_1(s)} + \int_0^{s_0} \frac{s\, ds}{\tilde{w}_0(s)}\right], \qquad s_0 \leq t < 1. \tag{1.6}$$

PROOF. Put

$$c = \int_0^{s_0} \frac{s\, ds}{\tilde{w}_0(s)},$$

$C_n = \{w(t) : w(t) \in C[s_0, 1],\ c(1-t) + 1/n \le w(t) \le c + 2/c + 1\}$, n is natural number. Define an operator as follows:

$$T_n w(t) = \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)\,ds}{w(s)} + (1-t)\left[\int_{s_0}^t \frac{s\,ds}{w(s)} + \int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)}\right] + \frac{1}{n}, \qquad w(t) \in C_n.$$

Notice $1 \ge \lambda + \lambda s + s \ge 0$, $s \in [s_0, 1]$ and

$$\begin{aligned} T_n w(t) &\le \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)}{c(1-s) + 1/n}\,ds + (1-t)\left[\int_{s_0}^t \frac{s\,ds}{c(1-s)+1/n} + \int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)}\right] + \frac{1}{n} \\ &\le \frac{1}{c}\int_t^1 (\lambda + \lambda s + s)\,ds + \int_{s_0}^t \frac{(1-t)s\,ds}{c(1-s)+1/n} + (1-t)\int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)} + \frac{1}{n} \le c + \frac{2}{c} + 1, \end{aligned}$$

we know that T_n is an operator from C_n into C_n. Similarly to the proof of Lemma 1.1, we have that T_n is continuous and compact. The Schauder fixed-point theorem tells us that there exists $w_n(t) \in C_n$ such that

$$w_n(t) = \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)\,ds}{w_n(s)} + (1-t)\left[\int_{s_0}^t \frac{s\,ds}{w_n(s)} + \int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)}\right] + \frac{1}{n}, \qquad s_0 \le t \le 1. \quad (1.7)$$

From (1.7), we have

$$w_n'(t) = \frac{-\lambda(1-t^2)}{w_n(t)} - \left(\int_{s_0}^t \frac{s\,ds}{w_n(s)} + \int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)}\right), \qquad s_0 \le t \le 1. \quad (1.8)$$

For fixed $b \in (s_0, 1)$, we have from (1.8)

$$|w_n'(t)| \le \frac{-2\lambda}{c} + \int_{s_0}^b \frac{s\,ds}{w_n(s)} + c \le \frac{-2\lambda}{c} + \frac{w_n(b)}{1-b} + c, \qquad s_0 \le t \le b,$$

i.e., $\{w_n'(t)\}$ is bounded on $[s_0, b]$. The Arzela-Ascoli theorem guarantees that there exists a subsequence of $\{w_n(t)\}$, which converges uniformly on $[s_0, b]$. Put $b = 1 - 1/k$, $k\ (> 1/(1-s_0))$ is a natural number, we may choose a subsequence $\{w_n^{(k)}(t)\}$ of $\{w_n(t)\}$, which converges uniformly on $[s_0, 1 - 1/k]$ and $\{w_n^{(k+1)}(t)\} \subseteq \{w_n^{(k)}(t)\}$ (for details, see [5]). Then the diagonal sequence $\{w_k^{(k)}(t)\}$ converges everywhere in $[s_0, 1)$ and it is easy to verify that $\{w_k^{(k)}(t)\}$ converges uniformly on any interval $[a, d] \subseteq [s_0, 1)$. Without loss of generality, let $\{w_k^{(k)}(t)\}$ be itself of $\{w_n(t)\}$, put $\tilde{w}_1(t) = \lim_{n\to+\infty} w_n(t)$ $(t \in [s_0, 1))$, then $\tilde{w}_1(t) \ge c(1-t)$, $t \in [s_0, 1)$ and $\tilde{w}_1(t)$ is continuous in $[s_0, 1)$.

Fixed $t \in [s_0, 1)$, we choose $\gamma \in (s_0, 1)$ to fit $t \in [s_0, \gamma]$ and the dominated functions $F_1(s) = 1/c$, $s \in [s_0, 1]$, $F_2(s) = 1/c(1-\gamma)$, $s \in [s_0, \gamma]$, equation (1.7) and the Lebesgue integral theorem lead to

$$\tilde{w}_1(t) = \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)\,ds}{\tilde{w}_1(s)} + (1-t)\left[\int_{s_0}^t \frac{s\,ds}{\tilde{w}_1(s)} + \int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)}\right], \qquad s_0 \le t < 1. \quad (1.9)$$

For $t \in [s_0, 1)$, we have from (1.9):

$$\tilde{w}_1(t) \le \frac{1}{c}\int_t^1 ds + (1-t)\left[\int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)} + \frac{1}{c}\int_{s_0}^t \frac{ds}{1-s}\right]. \quad (1.10)$$

From (1.10), we obtain $(t \in [s_0, 1))$:

$$\tilde{w}_1(t) \le (1-t)\left[\int_0^{s_0} \frac{s\,ds}{\tilde{w}_0(s)} + \frac{1}{c}(1 - \ln(1-t) + \ln(1-s_0))\right]. \quad (1.11)$$

Since $\lim_{t\to1^-}(1-t)\ln(1-t) = 0$, we have by (1.11) $\tilde{w}_1(1^-) = 0$ and

$$\int_{s_0}^1 \frac{ds}{\tilde{w}_1(s)} = +\infty.$$

This completes the proof.

2. EXISTENCE RESULTS

THEOREM 2.1. *For any fixed $\lambda \in (-1/2, 0)$, the problem $(*)$ has a solution $f(\eta) \in C^1[0, +\infty)$ at least, which satisfies*

$$0 < f'(\eta; \lambda) < 1, \qquad \eta > 0.$$

PROOF. Let $\tilde{w}_0(t)$ be a solution of (1.5) and $\tilde{w}_1(t)$ be a solution of (1.6). Put the function

$$g(t) = \begin{cases} \displaystyle\int_0^t \frac{ds}{\tilde{w}_0(s)}, & 0 \le t \le s_0, \\ \displaystyle\int_0^{s_0} \frac{ds}{\tilde{w}_0(s)} + \int_{s_0}^t \frac{ds}{\tilde{w}_1(s)}, & s_0 < t < 1. \end{cases} \tag{2.1}$$

Then $g(t)$ is strictly increasing in $[0, 1)$, $g(0) = 0$, $g(1^-) = +\infty$. Let $t = h(\eta)$ be the inverse function to $\eta = g(t)$ $(0 \le t < 1)$, $\eta_0 = \int_0^{s_0} ds/\tilde{w}_0(s)$, $s_0 = h(\eta_0)$. Define the function

$$f(\eta) = \int_0^\eta h(s)\, ds, \qquad 0 \le \eta < +\infty.$$

Next, we prove that $f(\eta)$ satisfies the problem $(*)$. Clearly, we have $f'(\eta) = h(\eta)$ and

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(+\infty) = 1.$$

From (2.1), we have

$$\eta = g[f'(\eta)] = \begin{cases} \displaystyle\int_0^{f'(\eta)} \frac{ds}{\tilde{w}_0(s)}, & 0 \le \eta \le \eta_0, \\ \displaystyle\int_0^{s_0} \frac{ds}{\tilde{w}_0(s)} + \int_{s_0}^{f'(\eta)} \frac{ds}{\tilde{w}_1(s)}, & \eta_0 < \eta < +\infty. \end{cases} \tag{2.2}$$

Differentiating (2.2) with respect to η, we have

$$f''(\eta) = \begin{cases} \tilde{w}_0(f'(\eta)), & 0 \le \eta < \eta_0, \\ \tilde{w}_1(f'(\eta)), & \eta_0 < \eta < +\infty. \end{cases} \tag{2.3}$$

Differentiating (1.5) and (1.6) with respect to t, we have

$$\tilde{w}_0'(t) = -\int_0^t \frac{s\, ds}{\tilde{w}_0(s)} - \frac{\lambda(1-t^2)}{\tilde{w}_0(t)}, \qquad 0 \le t < s_0, \tag{2.4}$$

and

$$\tilde{w}_1'(t) = -\left[\int_0^{s_0} \frac{s\, ds}{\tilde{w}_0(s)} + \int_{s_0}^t \frac{s\, ds}{\tilde{w}_1(s)}\right] - \frac{\lambda(1-t^2)}{\tilde{w}_1(t)}, \qquad s_0 < t < 1. \tag{2.5}$$

Inserting $t = f'(\eta)$, $s_0 = f'(\eta_0)$ into (2.4),(2.5) and using (2.3), we have

$$\tilde{w}_0'(f'(\eta)) = -\int_0^{f'(\eta)} \frac{s\, ds}{\tilde{w}_0(s)} - \frac{\lambda(1-f'^2(\eta))}{f''(\eta)} = -f(\eta) - \frac{\lambda(1-f'^2(\eta))}{f''(\eta)}, \qquad 0 < \eta < \eta_0, \tag{2.6}$$

and

$$\tilde{w}_1'(f'(\eta)) = -\left[\int_0^{f'(\eta_0)} \frac{s\, ds}{\tilde{w}_0(s)} + \int_{f'(\eta_0)}^{f'(\eta)} \frac{s\, ds}{\tilde{w}_1(s)}\right] - \frac{\lambda(1-f'^2(\eta))}{f''(\eta)}, \qquad \eta_0 < \eta < +\infty. \tag{2.7}$$

Put

$$k(\eta) = \int_0^{f'(\eta_0)} \frac{s\, ds}{\tilde{w}_0(s)} + \int_{f'(\eta_0)}^{f'(\eta)} \frac{s\, ds}{\tilde{w}_1(s)},$$

then $k'(\eta) = f'(\eta)$, $\eta_0 < \eta < +\infty$, $k(\eta_0) = \int_0^{f'(\eta_0)} s\, ds/\tilde{w}_0(s) = f(\eta_0)$, which imply $k(\eta) = f(\eta)$, $\eta_0 < \eta < +\infty$. Hence, we have from (2.7):

$$\tilde{w}_1'(f'(\eta)) = -f(\eta) - \frac{\lambda(1-f'^2(\eta))}{f''(\eta)}, \qquad \eta_0 < \eta < +\infty. \tag{2.8}$$

Differentiating (2.3) with respect to η and using (2.6) and (2.8), we have

$$f'''(\eta) = \begin{cases} \tilde{w}_0'(f'(\eta)) f''(\eta) = -f(\eta) f''(\eta) - \lambda(1 - f'^2(\eta)), & 0 < \eta < \eta_0, \\ \tilde{w}_1'(f'(\eta)) f''(\eta) = -f(\eta) f''(\eta) - \lambda(1 - f'^2(\eta)), & \eta_0 < \eta < +\infty. \end{cases}$$

This completes the proof.

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